Labelled Sequent Calculi

Lecture 1: The basics

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Plan

Lecture 1: The basics

- Modal logics
- Sequent calculus for classical and modal logics
- A labelled calculus for K (**labK**)

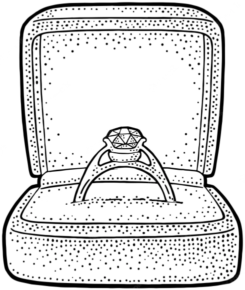
Lecture 2: The labelled approach

- Soundness and completeness for **labK**
- Rules for frame conditions: a general recipe
- Countermodels and termination

Lecture 3: Beyond the modal cube

- Neighbourhood semantics for conditional logics
- (Bi-)Relational semantics for intuitionistic (modal) logics

Modal logics



The S5 cube of modal logics

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nec	if A is provable, so is $\Box A$
k	$\Box(A \to B) \to (\Box A \to \Box B)$

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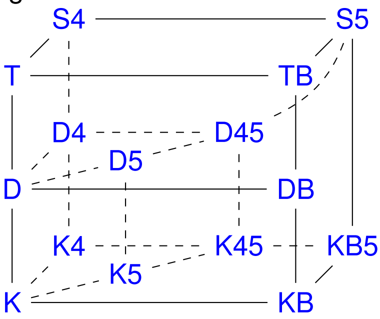

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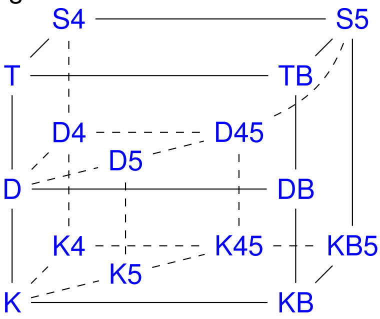

☞ $\vdash_{CP} A$ $\rightsquigarrow$ A is derivable from the axioms of CP

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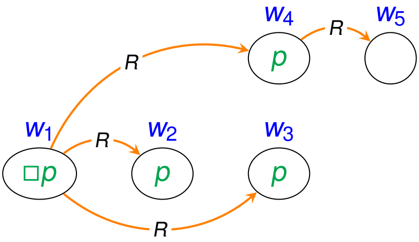
Kripke models for K

$$\mathcal{M} = \langle W, R, v \rangle$$



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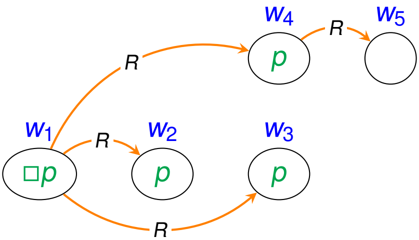

☞ Satisfiability $\mathcal{M}, w \Vdash A$

$\mathcal{M}, w \Vdash \Box A$ *iff* for all y s.t. xRy, $y \Vdash A$

$\mathcal{M}, w \Vdash \Diamond A$ *iff* there exists y s.t. xRy and $y \Vdash A$

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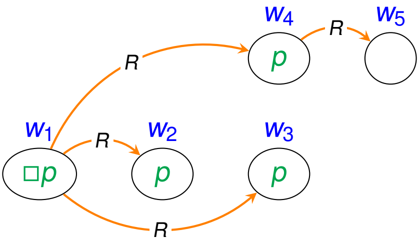

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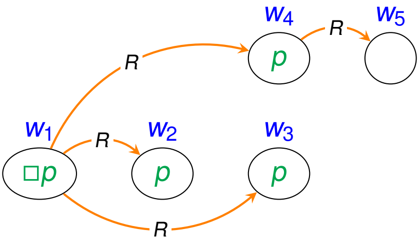

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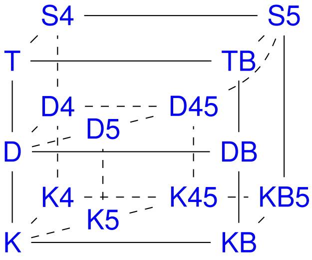
Soundness and Completeness. $\vdash_{\mathsf{K}} A$ iff $\models_{\mathcal{K}} A$ [BdRV, 2001]

Kripke models for the S5 cube



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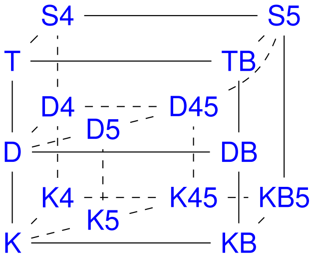

d	$\Box A \to \Diamond A$	Ser.	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Refl.	$\forall x (xRx)$
b	$A \to \Box\Diamond A$	Sym.	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Tran.	$\forall x \forall y \forall z (xRy \land yRz \to xRz)$
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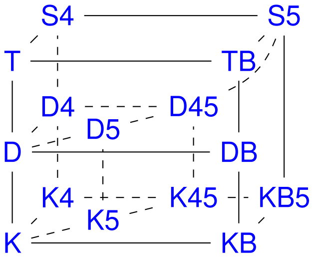

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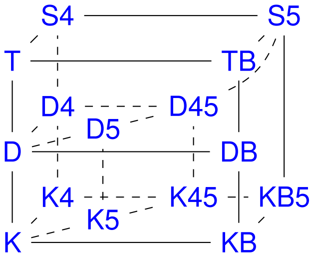

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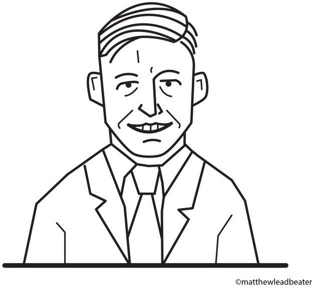
Exercise. Prove that Ref+Sym+Tr iff Ref+Eucl.

Sequent calculus
for classical and modal logics



Gentzen-style calculus for classical logic

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Formula interpretation $\quad i(\Gamma \Rightarrow \Delta) \; := \; \bigwedge \Gamma \to \bigvee \Delta$

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G3cp

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \qquad\qquad \bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta}$$

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$$\wedge_{\mathrm{R}}\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad\qquad \wedge_{\mathrm{L}}\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

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☞ $\vdash_{\textbf{G3cp}} \Gamma \Rightarrow \Delta \quad \rightsquigarrow \quad$ there is a derivation of $\Gamma \Rightarrow \Delta$ in **G3cp**

Example

$$
\to_R \dfrac{\land_L \dfrac{\to_R \dfrac{\to_L \dfrac{\text{init} \dfrac{}{q \to r, p \Rightarrow r, p} \qquad \to_L \dfrac{\text{init} \dfrac{}{q, p \Rightarrow r, q} \qquad \text{init} \dfrac{}{q, r, p \Rightarrow r}}{q, q \to r, p \Rightarrow r}}{p \to q, q \to r, p \Rightarrow r}}{p \to q, q \to r \Rightarrow p \to r}}{(p \to q) \land (q \to r) \Rightarrow p \to r}}{\Rightarrow ((p \to q) \land (q \to r)) \to (p \to r)}
$$

Main results

Soundness. If $\vdash_{\mathbf{G3cp}} \Rightarrow A$ then $\vdash_{\mathrm{CP}} A$.

Completeness. If $\vdash_{\mathrm{CP}} A$ then $\vdash_{\mathbf{G3cp}} \Rightarrow A$.

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$$\mathrm{MP}\,\frac{\vdash_{\mathrm{CP}} A \quad A \vdash_{\mathrm{CP}} B}{\vdash_{\mathrm{CP}} B} \quad \rightsquigarrow \quad \mathrm{cut}\,\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

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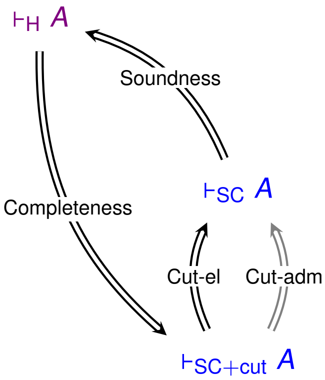
Cut-admissibility. If $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta, A$ and $\vdash_{\mathbf{G3cp}} A, \Gamma' \Rightarrow \Delta'$, then $\vdash_{\mathbf{G3cp}} \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Main results, graphically



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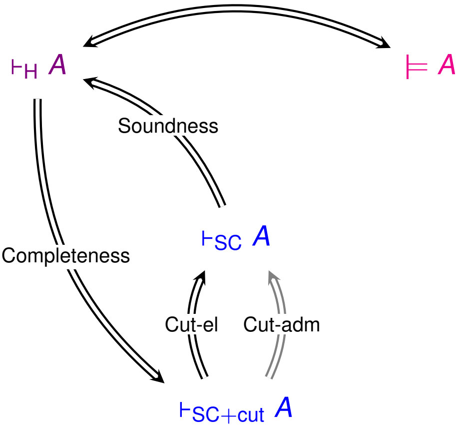

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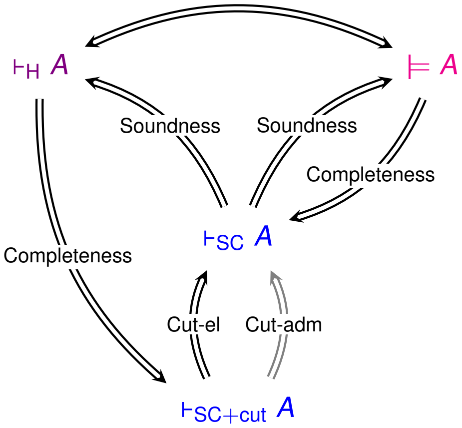

Gentzen-style sequent calculi for modal logics

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[Fitting, 1983], [Takano, 1992]

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$$\text{k}\ \frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta}$$

- Sequent calculus for K: **G3cp** $+$ k

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- Sequent calculus for K: **G3cp** $+$ k

Invertibility of a rule. If the conclusion of the rule is derivable, then each of its premisses is derivable.

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- Sequent calculus for K: **G3cp** + k
- Sequent calculus for T: **G3cp** + k + t

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- Sequent calculus for K: **G3cp** $+$ k
- Sequent calculus for T: **G3cp** $+$ k $+$ t
- Sequent calculus for S4: **G3cp** $+$ 4 $+$ t

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$$4\,\frac{\Box B_1, \ldots, \Box B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \quad 45\,\frac{\Box B_1, \ldots, \Box B_n \Rightarrow A, \Box C_1, \ldots, \Box C_m}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Box C_1, \ldots, \Box C_m, \Delta}$$

- Sequent calculus for K: **G3cp** $+$ k
- Sequent calculus for T: **G3cp** $+$ k $+$ t
- Sequent calculus for S4: **G3cp** $+$ 4 $+$ t
- Sequent calculus for S5: **G3cp** $+$ 45 $+$ t

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But..

The sequent calculus for S5 is NOT cut-free complete.

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Analyticity. All formulas in a derivation are subformulas of the formula at the root.

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☞ "*A cut-free sequent calculus for* S5 *will require additional machinery in the rule format or a very different, possibly semantic, proof of cut admissibility.*" [Lellmann & Pattinson, 2013]

A syntactic solution

Enrich the structure of sequents

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- ☞ Hypersequents

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$$\mathsf{k}_\mathsf{L}^H \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \mathsf{k}_\mathsf{R}^H \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

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☞ Hypersequents

For S5 [Avron, 1995], ...

$$i(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) \;:=\; \bigvee_{i=1}^{n} \Box(i(\Gamma_i \Rightarrow \Delta_i))$$

$$\mathsf{k}^H_\mathsf{L}\;\frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \mathsf{k}^H_\mathsf{R}\;\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

☞ Nested sequents

[Kashima, 1994], [Brünnler, 2009] [Poggiolesi, 2009], ...

$$i(\Gamma \Rightarrow \Delta, [\Sigma_0], \ldots, [\Sigma_k]) \;:=\; \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigvee_{i=0}^{n} \Box(i(\Sigma_i))$$

A syntactic solution

Enrich the structure of sequents

- Hypersequents

  For S5 [Avron, 1995], ...

$$i(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) := \bigvee_{i=1}^{n} \Box(i(\Gamma_i \Rightarrow \Delta_i))$$

$$\mathsf{k}^H_L \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \mathsf{k}^H_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

- Nested sequents

  [Kashima, 1994], [Brünnler, 2009] [Poggiolesi, 2009], ...

$$i(\Gamma \Rightarrow \Delta, [\Sigma_0], \ldots, [\Sigma_k]) := \bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{i=0}^{n} \Box(i(\Sigma_i))$$

$$\Box_L \frac{S\{\, \Box A, \Gamma \Rightarrow \Delta, [A, \Gamma' \Rightarrow \Delta'] \,\}}{S\{\, \Box A, \Gamma \Rightarrow \Delta, [\Gamma' \Rightarrow \Delta'] \,\}} \qquad \Box_R \frac{S\{\, \Gamma \Rightarrow \Delta, [\Rightarrow A] \,\}}{S\{\, \Gamma \Rightarrow \Delta, \Box A \,\}}$$

A semantic solution

Enrich the language of the calculus

A semantic solution

Enrich the language of the calculus

☞ Labelled sequent calculi

A semantic solution

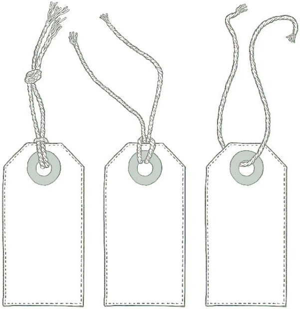
Enrich the language of the calculus

☞ Labelled sequent calculi

- [Kanger, 1957] Spotted formulas for S5
- [Fitting, 1983], [Goré 1998] Tableaux + labels
- [Simpson, 1994], [Viganò, 1998] Natural deduction + labels
- [Mints, 1997], [Viganò, 2000] Sequent calculus + labels
- ...

- [Negri, 2005], [Negri, 2003], ...

Labelled calculus for K



Enriching the language

$$A, B ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

$$\neg A := A \to \bot$$

$$\Diamond A \leftrightarrow \neg \Box \neg A$$

$$\Box A \leftrightarrow \neg \Diamond \neg A$$

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☞ Countably many variables $x, y, z, \ldots$ (labels)

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☞ Countably many variables $x, y, z, \ldots$ (labels)

☞ Labelled formulas

- $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
- $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)

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- Labelled sequent

$$\mathcal{R}, \Gamma \Rightarrow \Delta$$

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$$\mathcal{R}, \Gamma \Rightarrow \Delta$$

$\mathcal{R}$ multiset of relational atoms, Γ, Δ multisets of labelled formulas

Enriching the language

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☞ Labelled formulas

- $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
- $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)

☞ Labelled sequent

$$\mathcal{R}, \Gamma \Rightarrow \Delta \qquad\qquad i(\mathcal{R}, \Gamma \Rightarrow \Delta) := ???$$

$\mathcal{R}$ multiset of relational atoms, Γ, Δ multisets of labelled formulas

Labelled calculus **labK** for K

$$\text{init}\ \frac{}{\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \text{init}\ \frac{}{\mathcal{R}, x : \bot, \Gamma \Rightarrow \Delta}$$

Labelled calculus **labK** for K

$$\text{init}\ \frac{}{\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \text{init}\ \frac{}{\mathcal{R}, x : \bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_R\ \frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B}$$

$$\vee_L\ \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \vee B}$$

$$\rightarrow_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R\ \frac{x : A, \mathcal{R}\Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \rightarrow B}$$

Labelled calculus **labK** for K

$$\text{init}\ \frac{}{\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \text{init}\ \frac{}{\mathcal{R}, x : \bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_R\ \frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B}$$

$$\vee_L\ \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \vee B}$$

$$\rightarrow_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R\ \frac{x : A, \mathcal{R}\Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \rightarrow B}$$

$$\Box_L\ \frac{xRy, \mathcal{R}, y : A, x : \Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ \text{(y!)}$$

$y!$ $\rightsquigarrow$ y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Labelled calculus **labK** for K

$$\text{init}\ \frac{}{\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p}
\qquad
\text{init}\ \frac{}{\mathcal{R}, x : \bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_{\mathsf{R}}\ \frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta}
\qquad
\wedge_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \vee B, \Gamma \Rightarrow \Delta}
\qquad
\vee_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \vee B}$$

$$\rightarrow_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \rightarrow B, \Gamma \Rightarrow \Delta}
\qquad
\rightarrow_{\mathsf{R}}\ \frac{x : A, \mathcal{R}\Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \rightarrow B}$$

$$\Box_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, y : A, x : \Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}
\qquad
\Box_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ (\mathrm{y!})$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : \Diamond A, \Gamma \Rightarrow \Delta}\ (\mathrm{y!})
\qquad
\Diamond_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A}$$

$y!$ $\rightsquigarrow$ y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Derivation example

$$
{\scriptstyle \to_{\mathrm{L}}}\;\dfrac{
{\scriptstyle \Diamond_{\mathrm{R}}}\;\dfrac{
{\scriptstyle \mathrm{init}}\;\dfrac{}{xRy, y : p \Rightarrow y : q, x : \Diamond p, y : p}
}{xRy, y : A \Rightarrow y : q, x : \Diamond p}
\qquad
{\scriptstyle \Box_{\mathrm{L}}}\;\dfrac{
{\scriptstyle \mathrm{init}}\;\dfrac{}{xRy, x : \Box q, y : q, y : p \Rightarrow y : q}
}{xRy, x : \Box q, y : p \Rightarrow y : q}
}{
{\scriptstyle \to_{\mathrm{R}}}\;\dfrac{xRy, x : \Diamond p \to \Box q, y : p \Rightarrow y : q}{
{\scriptstyle \Box_{\mathrm{R}}}\;\dfrac{xRy, x : \Diamond p \to \Box q \Rightarrow y : p \to q}{
{\scriptstyle \to_{\mathrm{R}}}\;\dfrac{x : \Diamond p \to \Box q \Rightarrow x : \Box(p \to q)}{\Rightarrow x : (\Diamond p \to \Box q) \to \Box(p \to q)}
}
}
}
$$

Derivation example

$$
\dfrac{\dfrac{\overline{xRy, y : p \Rightarrow y : q, x : \Diamond p, y : p}\ \text{init}}{xRy, y : A \Rightarrow y : q, x : \Diamond p}\ \Diamond_R \qquad \dfrac{\overline{xRy, x : \Box q, y : q, y : p \Rightarrow y : q}\ \text{init}}{xRy, x : \Box q, y : p \Rightarrow y : q}\ \Box_L}{\dfrac{xRy, x : \Diamond p \to \Box q, y : p \Rightarrow y : q}{\dfrac{xRy, x : \Diamond p \to \Box q \Rightarrow y : p \to q}{\dfrac{x : \Diamond p \to \Box q \Rightarrow x : \Box(p \to q)}{\Rightarrow x : (\Diamond p \to \Box q) \to \Box(p \to q)}\ \to_R}\ \Box_R}\ \to_R}\ \to_L
$$

Exercise. Construct a derivation of the following:

$$\Rightarrow x : \Box(p \to q) \to (\Box p \to \Box q)$$
$$\Rightarrow x : \Box(p \to q) \to (\Diamond p \to \Diamond q)$$
$$\Rightarrow x : \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$$

Summing up

	G3cp	G3cp+ modal r.	Labelled	Nested
☞ Formula interpretation	yes	yes	no	yes
☞ Analyticity	yes	no	subterm	yes
☞ Termination	yes	yes	yes	yes
☞ Invertibility	yes	no	yes	yes
☞ Modularity	n.a.	no	yes*	yes

* Even beyond the S5 cube!

Main references

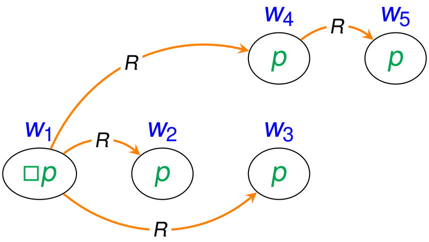
- Blackburn, de Rijke and Venema, Modal Logic, Cambridge University Press, 2001.
- Gentzen, Untersuchungen über das logische schliessen, Matematische Zeitschrift, 39, 1934-35.
- Lellmann and Dirk Pattinson, Correspondence between modal Hilbert axioms and sequent rules with an application to S5, IJCAR 2013, Springer, 2013.
- Negri, Contraction-free sequent calculi for geometric theories with an application to Barr's theorem, Arch. Math. Logic 42, 2003.
- Negri, Proof analysis in modal logic, Journal of Philosophical Logic 34.5, 2005.
- Takano, Subformula property as a substitute for cut-elimination in modal propositional logics, Mathematica japonica, 37, 1992.
- Troelstra and Schwichtenberg, Basic proof theory, vol. 43, Cambridge University Press, 2000.

Appendix

Kripke models for K4

$$\mathcal{M} = \langle W, R, v \rangle$$

$4: \ \Box A \supset \Box\Box A$ Transitivity of R



Soundness and Completeness. $\vdash_{K4} A$ iff $\models_{\mathcal{K}4} A$

Soundness of the k rule

$$\text{k}\ \frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta}$$

Properties of sequent calculus

- Analyticity $\rightsquigarrow$ All formulas in a derivation are subformulas of the formula at the root.

$$\text{cut}\,\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

- Context-independence $\rightsquigarrow$ Rules modify only the principal formula
- Termination $\rightsquigarrow$ Decision procedure
- Invertibility of the rules $\rightsquigarrow$ Root-first proof search without choices, Countermodel construction
- Modularity over the S5 cube $\rightsquigarrow$ Adding rules in correspondence to frame conditions

A “syntactic” solution: hypersequents

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

$$i(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) := \bigvee_{i=1}^{n} \Box(i(\Gamma_i \Rightarrow \Delta_i))$$

☞ Hypersequent rules for S5

$$\mathsf{k}^H_\mathsf{L}\ \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \mathsf{k}^H_\mathsf{R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

$$\mathsf{k}_\mathsf{rf}\ \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta}$$

Exercise. Prove soundness of the k^H_R rule.

☞ Hypersequents for the S5 cube

Nested sequents

$$\Gamma \Rightarrow \Delta, [\Sigma_1], \ldots, [\Sigma_k]$$

for $\Sigma_1, \ldots, \Sigma_n$ nested sequents

$$i(\Gamma \Rightarrow \Delta, [\Sigma_0], \ldots, [\Sigma_k]) := \bigwedge \Gamma \supset \bigvee \Delta \vee \bigvee_{i=0}^{n} \Box(i(\Sigma_i))$$

☞ The nested rules for $\Box$

$$\Box_L \frac{S\{\, \Box A, \Gamma \Rightarrow \Delta, [A, \Gamma' \Rightarrow \Delta'] \,\}}{S\{\, \Box A, \Gamma \Rightarrow \Delta, [\Gamma' \Rightarrow \Delta'] \,\}} \qquad \Box_R \frac{S\{\, \Gamma \Rightarrow \Delta, [\Rightarrow A] \,\}}{S\{\, \Gamma \Rightarrow \Delta, \Box A \,\}}$$

Exercise. Show soundness of the k_R^H rule.

☞ Full modularity over the S5 cube