Labelled Sequent Calculi

Lecture 2: The labelled approach

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Plan

Lecture 1: The basics

- Modal logics
- Sequent calculus for classical and modal logics
- A labelled calculus for K (**labK**)

Lecture 2: The labelled approach

- Soundness and completeness for **labK**
- Rules for frame conditions: a general recipe
- Countermodels and termination

Lecture 3: Beyond the modal cube

- Neighbourhood semantics for conditional logics
- (Bi-)Relational semantics for intuitionistic (modal) logics

Recap: **labK**

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- ☞ Countably many variables $x, y, z, \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ xRy $\rightsquigarrow$ "x has access to y" (relational atoms)
  - ▹ $x : A$ $\rightsquigarrow$ "x satisfies A" (labelled formulas)

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- Rules for $\Box$ and $\Diamond$

$$\Box_L \frac{xRy, \mathcal{R}, y : A, x : \Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} \text{ (y!)}$$

$$\Diamond_L \frac{xRy, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : \Diamond A, \Gamma \Rightarrow \Delta} \text{ (y!)} \qquad \Diamond_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A}$$

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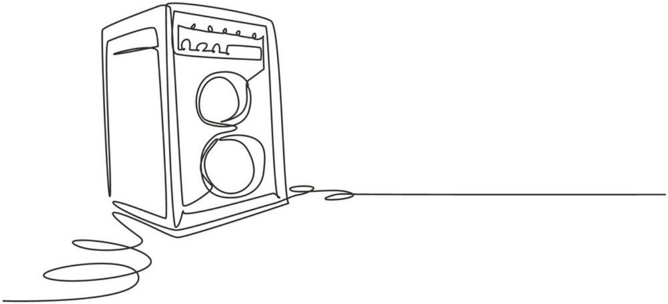
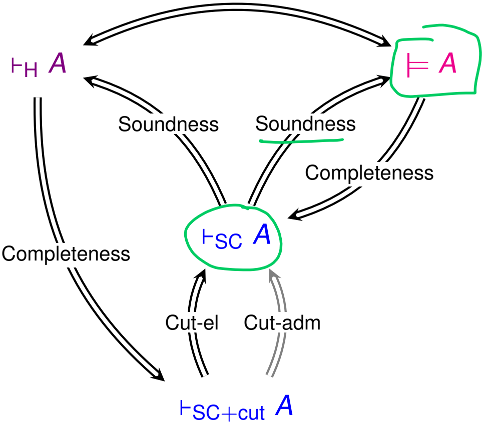
- $\vdash_{\mathbf{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \rightsquigarrow \quad \mathcal{R}, \Gamma \Rightarrow \Delta$ is derivable in **labK**

Soundness and completeness for **labK**



Main results, graphically



Validity of sequents

Given a sequent $S = \underbrace{\mathcal{R}, \Gamma \Rightarrow \Delta}$, and a model $\underbrace{\mathcal{M} = \langle W, R, v\rangle}$, let $\underbrace{\mathrm{Lb(S)}} = \{x \mid x \in \mathcal{R} \cup \Gamma \cup \Delta\}$, and $\rho : \mathrm{Lb(S)} \to W$ (interpretation)

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☞ Satisfiability of labelled formulas at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash xRy \quad \textit{iff} \quad \mathcal{M} \Vdash \rho(x) R \rho(y)$$
$$\mathcal{M}, \rho \Vdash x : A \quad \textit{iff} \quad \mathcal{M}, \rho(x) \Vdash A$$

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☞ Satisfiability of sequents at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash \underline{\mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{iff}$$

$$\left(\begin{array}{c} \underline{\text{if}} \text{ for all } \underline{xRy \in \mathcal{R}}, \underline{x : G \in \Gamma}, \mathcal{M}, \rho \Vdash xRy \text{ and } \mathcal{M}, \rho \Vdash x : G \\ \underline{\text{then}} \text{ for some } x : D \in \Delta,\ \mathcal{M}, \rho \Vdash x : D \end{array} \right)$$

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☞ Satisfiability of sequents at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta \quad \text{iff}$$

if for all $xRy \in \mathcal{R}, x : G \in \Gamma$, $\mathcal{M}, \rho \Vdash xRy$ and $\mathcal{M}, \rho \Vdash x : G$

then for some $x : D \in \Delta$, $\mathcal{M}, \rho \Vdash x : D$

☞ Validity of sequents in a class of frames $\mathcal{X}$: K

$$\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \text{iff} \quad \text{for any } \rho \text{ and any } \mathcal{M} \in \mathcal{X},\ \mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta$$

Soundness of **labK**

Soundness. If $\vdash_{\mathbf{labK}} \underset{\sim\sim\sim}{\overset{\varnothing}{\Rightarrow x : A}}$ then $\models_{\mathcal{K}} \Rightarrow x : A$

Proof. Induction on the height h of the derivation. — # of nodes occurring in its longest branch, − 1

$h = 0$

$h > 0$

$$\frac{P:\quad xR^{t}y, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{C:\quad \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ \Box_R$$

$\underline{\models_{\mathcal{K}} P}$ but $\mathcal{M}, \rho \not\Vdash C$

$\mathcal{M}, \rho \Vdash$ all formulas in $\mathcal{R}, \Gamma$

$\mathcal{M}, \rho \not\Vdash$ any formulas in Δ

$\mathcal{M}, \rho \not\Vdash x : \Box A \leadsto \mathcal{M}, \rho(x) \not\Vdash \Box A$

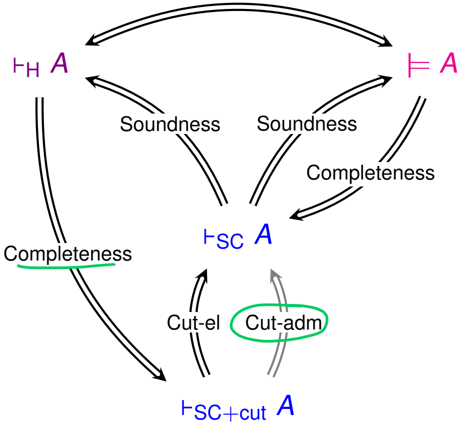
$\exists y$ s.t. $\rho(x)\, \underline{R}\, y$ and $\underline{y \not\Vdash A}$

$\rho' = \rho$

$\rho'(y) = y$

$\mathcal{M}, \rho' \not\Vdash P$

Main results, graphically



Towards cut-admissibility 1/3

⇒ Negri, 2005
⇒ Structural Proof Theory
2001

Towards cut-admissibility 1/3

- Substitution on labelled formulas:

$$\begin{aligned} xRy[z/y] &\quad := \quad xRz \\ y : A[z/y] &\quad := \quad z : A \end{aligned}$$

- Substitution on multisets of labelled formulas $\Gamma[z/y]$

Towards cut-admissibility 1/3

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Height-preserving admissibility of substitution.

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}[y/x], \Gamma[y/x] \Rightarrow \Delta[y/x]}$$

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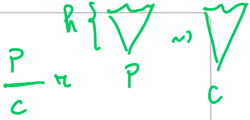

Height-preserving admissibility of substitution.

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}[y/x], \Gamma[y/x] \Rightarrow \Delta[y/x]}$$

Height-preserving admissibility of weakening. (ϕ is xRy or $x : A$)

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\phi, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, \phi}$$

Towards cut-admissibility 2/3

Invertibility.
For every r, if the conclusion of r is derivable with a derivation of height h, then each of its premisses is derivable, with at most the same h.

Towards cut-admissibility 2/3

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Proof. Case of $\Box_R$:

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}$$

Towards cut-admissibility 2/3

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For every r, if the conclusion of r is derivable with a derivation of height h, then each of its premisses is derivable, with at most the same h.

Proof. Case of $\Box_R$:

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}$$

By induction on the height h of the derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A$.

Towards cut-admissibility 2/3

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For every r, if the conclusion of r is derivable with a derivation of height h, then each of its premisses is derivable, with at most the same h.

Proof. Case of $\Box_R$:

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}$$

By induction on the height h of the derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A$.

- If $h = 0$, then $xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A$ is derivable.

Towards cut-admissibility 2/3

Invertibility.
For every r, if the conclusion of r is derivable with a derivation of height h, then each of its premisses is derivable, with at most the same h.

Proof. Case of $\Box_R$:

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}$$

By induction on the height h of the derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A$.

- If $h = 0$, then $xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A$ is derivable.
- If $h > 0$, two cases:

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} \qquad \mathrm{r} \frac{\mathcal{R}', \Gamma' \Rightarrow \Delta', x : \Box A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}$$

IH

and then R again

Towards cut-admissibility 3/3

Height-preserving admissibility of contraction. (ϕ is xRy or $x : A$)

$x : \Box A$

$$\frac{\phi, \phi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\phi, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, \phi, \phi}{\mathcal{R}, \Gamma \Rightarrow \Delta, \phi}$$

$x : \Box A$

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Proof. Case $\phi = x : \Box A$

Assume

$$\frac{\vdots}{\dfrac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A, x : \Box A}\ \text{R} = \Box R}$$

inv

$$\frac{\dfrac{xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A, y : A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A}\ \Box R$$

Cut admissibility

Admissibility of cut.

$$\text{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad x : A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $(w, h_1 + h_2)$.

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$$\mathrm{cut}\ \dfrac{\Box_R\ \dfrac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : \Box A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} \qquad \Box_L\ \dfrac{xRz, \mathcal{R}', x : \Box A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}', x : \Box A, \Gamma' \Rightarrow \Delta'}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut admissibility

Admissibility of cut.

$$\text{cut}\ \dfrac{\mathcal{R}, \Gamma \Rightarrow \Delta, {\color{red} x : A} \qquad {\color{red} x : A}, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $({\color{red} w}, {\color{blue} h_1 + h_2})$.

$$\text{cut}\ \dfrac{\Box_R\ \dfrac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : \cancel{\Box}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, {\color{red} x : \Box A}} \qquad \Box_L\ \dfrac{xRz, \mathcal{R}', x : \Box A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}', {\color{red} x : \Box A}, \Gamma' \Rightarrow \Delta'}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

z/y

$$\text{ctr}\ \dfrac{\text{cut}\ \dfrac{xRz, \mathcal{R}, \Gamma \Rightarrow \Delta, \underline{z : A} \qquad \text{cut}\ \dfrac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A \qquad xRz, \mathcal{R}', x : \Box A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}, \mathcal{R}', \underline{z : A}, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{\mathcal{R}, \mathcal{R}, xRz, xRz, \mathcal{R}', \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}}{\underline{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}$$

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Admissibility of cut.

$$\mathrm{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \qquad x : A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $(w, h_1 + h_2)$.

$$\mathrm{cut}\ \frac{\square_R \dfrac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : \square A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \square A} \qquad \square_L \dfrac{xRz, \mathcal{R}', x : \square A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}', x : \square A, \Gamma' \Rightarrow \Delta'}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

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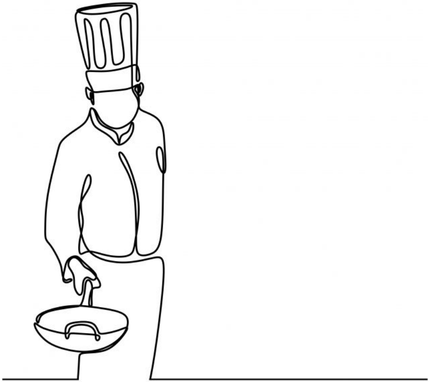
Cut-free completeness. If $\vdash_{\mathsf{K}} A$ then $\vdash_{\mathbf{labK}} \Rightarrow x : A$.

Frame conditions: a general recipe



From frame conditions to rules

Name	Axiom	Frame condition	
d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Reflexivity	$\forall x (xRx)$
b	$A \to \Box\Diamond A$	Symmetry	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Transitivity	$\forall x \forall y \forall z (xRy \wedge yRz \to xRz)$
5	$\Diamond A \to \Box\Diamond A$	Euclideaness	$\forall x \forall y \forall z (xRy \wedge xRz \to yRz)$

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☞ Frame conditions as first order logic formulas

$$
\begin{array}{rcl}
t & ::= & x \\
A, B & ::= & \underline{xRy} \mid A \wedge B \mid A \vee B \mid A \to B \mid \forall x A \mid \exists x A
\end{array}
$$

From (geometric) axioms to rules

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$$\forall \vec{x}\Big((P_1 \wedge \ldots \wedge P_n) \rightarrow \bigvee_{i=1}^{m} \exists \vec{y}_i \big(Q_{i1} \wedge \ldots \wedge Q_{ik_i}\big)\Big)$$

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  - $\vec{x}$, $\vec{y}_i$ are (possibly empty) vectors of variables
  - $n, m \geq 0, \quad k_1, \ldots, k_m \geq 1$

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- $P_1, \ldots, P_n$, $Q_{i1}, \ldots, Q_{ik_i}$ atomic formulas

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☞ Labelled rule

$$\mathsf{r}\,\frac{\Xi_1[\vec{z_1}/\vec{y_1}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z_m}/\vec{y_m}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \underline{\mathcal{R}}, \underline{\Gamma} \Rightarrow \underline{\Delta}}$$

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- $\Pi = \{P_1, \ldots, P_n\}$ and $\Xi_i = \{Q_{i1}, \ldots, Q_{ik_i}\}$ are multisets

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$$\mathrm{r}\ \frac{\Xi_1[\vec{z_1}/\vec{y_1}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z_m}/\vec{y_m}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

- $\Pi = \{P_1, \ldots, P_n\}$ and $\Xi_i = \{Q_{i1}, \ldots, Q_{ik_i}\}$ are multisets
- $\Xi[\vec{z}/\vec{y}]$: multiset obtained by substituting the free variables $\vec{y}$ with variables $\vec{z}$ in every formula of Ξ

From (geometric) axioms to rules

☞ Geometric axioms [Simpson, 1994], [Negri, 2003]

$$\forall \vec{x}\Big((P_1 \wedge \ldots \wedge P_n) \to \bigvee_{i=1}^{m} \exists \vec{y}_i \big(Q_{i1} \wedge \ldots \wedge Q_{ik_i}\big)\Big)$$

- $\vec{x}$, $\vec{y}_i$ are (possibly empty) vectors of variables
- $n, m \geq 0, \quad k_1, \ldots, k_m \geq 1$
- $P_1, \ldots, P_n$, $Q_{i1}, \ldots, Q_{ik_i}$ atomic formulas
- $\vec{y}_1, \ldots, \vec{y}_m$ do not occur in any of $P_1, \ldots, P_n$

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$$r\,\frac{\Xi_1[\vec{z_1}/\vec{y_1}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z_m}/\vec{y_m}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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- $\Xi[\vec{z}/\vec{y}]$: multiset obtained by substituting the free variables $\vec{y}$ with variables $\vec{z}$ in every formula of Ξ
- $\vec{z_1}, \ldots, \vec{z_m}$ do not occur in $\mathcal{R}, \Gamma \cup \Delta$

Examples

$$\forall \vec{x}\Big((P_1 \wedge \ldots \wedge P_n) \to \bigvee_{i=1}^{m} \exists \vec{y}_i \big(Q_{i1} \wedge \ldots \wedge Q_{ik_i}\big)\Big)$$

$$\downarrow$$

$$\text{r}\ \frac{\Xi_1[\vec{z_1}/\vec{y_1}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z_m}/\vec{y_m}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

- ☞ Seriality $\forall x \exists \underline{y}(xRy)$ $\quad \forall x\,(\top \to \exists y\,(xRy))$

$$\frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\,(y!)$$

- ☞ Reflexivity $\forall x(xRx)$

$$\frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}$$

- ☞ Transitivity $\forall x \forall y \forall z(\underline{xRy \wedge yRz} \to xRz)$

$$\frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled calculi for extensions of K

Rules for **labK**, plus structural rules for frame conditions:

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ (y!) \qquad \text{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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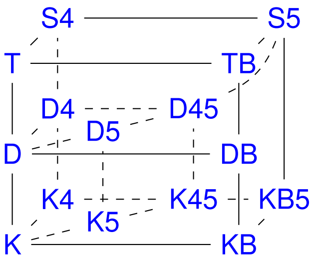
☞ **labX**: labelled sequent calculi for logics in the S5 cube



Labelled calculi for extensions of K

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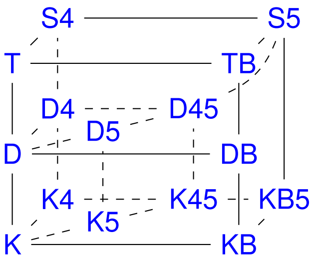
- **labX**: labelled sequent calculi for logics in the S5 cube
- $\vdash_{\textbf{labX}} A \ \rightsquigarrow\ A$ is derivable in **labX**



Adequacy of **labX**

Soundness. For all the logics in the S5 cube,
If $\vdash_{\mathbf{labX}} \Rightarrow x : A$ then $\models_{\mathcal{X}} \Rightarrow x : A$

Example. If the premiss of rule Ser is valid, then its conclusion is valid in all serial frames.

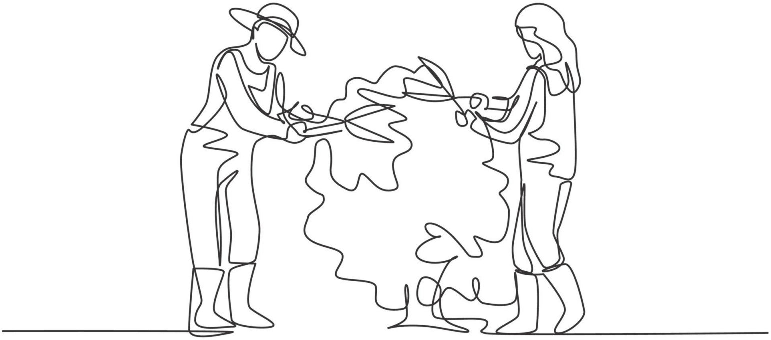
$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ (y!)$$

Admissibility of cut. Cut is admissible in **labX**:

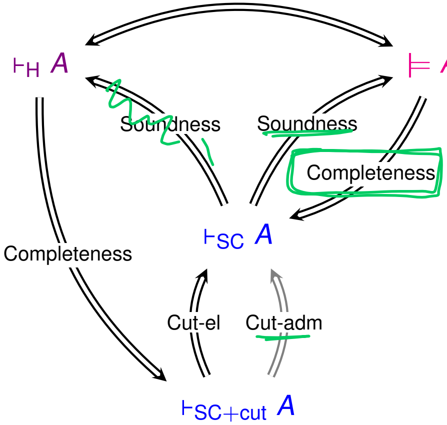
$$\text{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad x : A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut-free completeness. For any logic X in the S5 cube,
If $\vdash_{X} A$ then $\vdash_{\mathbf{labX}} \Rightarrow x : A$.

Countermodels and termination



Main results, graphically



A semantic proof of completeness

Cut-free completeness (semantically). For any logic X in the S5 cube,
If $\models_{\mathcal{X}} A$ then $\vdash_{\mathbf{labX}} \Rightarrow x : A$.

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That is, we construct a model $\mathcal{M}^{\times}$ with frame conditions $\mathcal{X}$ and a
interpretation
~~realisation~~ $\rho^{\times}$ such that $\mathcal{M}^{\times}, \rho^{\times} \not\models\ \Rightarrow x : A$.

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Therefore $\mathcal{M}^{\times}, \rho^{\times}(x) \not\models A$ and $\not\models_{\mathcal{X}} A$.

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- Infinite search tree $\rightsquigarrow$ Infinite countermodel
- Finite search tree $\rightsquigarrow$ Finite countermodel

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