Labelled Sequent Calculi

Lecture 3: Beyond the modal cube

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Plan

Lecture 1: The basics

- Modal logics
- Sequent calculus for classical and modal logics
- A labelled calculus for K (**labK**)

Lecture 2: The labelled approach

- Soundness and completeness for **labK**
- Rules for frame conditions: a general recipe
- Countermodels and termination

Lecture 3: Beyond the modal cube

- Neighbourhood semantics for conditional logics
- (Bi-)Relational semantics for intuitionistic (modal) logics

Main references from lecture 2

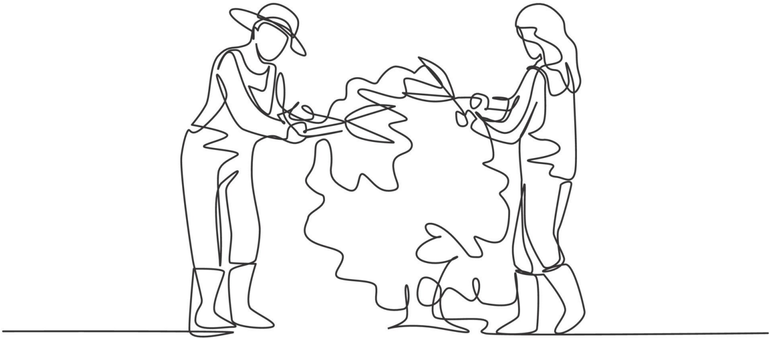
Labelled calculi for modal logics

- Negri, Proof analysis in modal logic, Journal of Philosophical Logic 34.5, 2005.
- Negri and von Plato, Structural proof theory, Cambridge University Press, 2008.

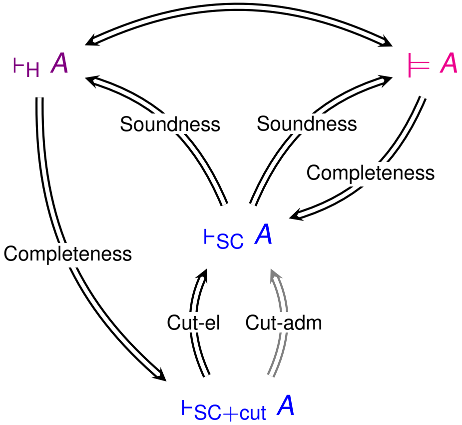
From geometric axioms to rules

- Negri, Contraction-free sequent calculi for geometric theories with an application to Barr's theorem, Arch. Math. Logic 42, 2003.
- Negri, Proof analysis beyond geometric theories: from rule systems to systems of rules, Journal of Logic and Computation 26.2, 2014.

Countermodels and termination



Main results, graphically



A semantic proof of completeness

Cut-free completeness (semantically). For any logic X in the S5 cube,
If $\models_{\mathcal{X}} A$ then $\vdash_{\mathbf{labX}} \Rightarrow x : A$.

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Therefore $\mathcal{M}^{\times}, \rho^{\times}(x) \not\models A$ and $\not\models_{\mathcal{X}} A$.

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- Infinite search tree $\rightsquigarrow$ Infinite countermodel

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- Infinite search tree $\rightsquigarrow$ Infinite countermodel
- Finite search tree $\rightsquigarrow$ Finite countermodel

Exhaustive (infinite) search tree

Given a sequent $\mathcal{R}_0, \Gamma_0 \Rightarrow \Delta_0$, construct a tree $\mathcal{T}$ of sequents by applying "macro-steps" of rules:

(Base) $S_0 = \mathcal{R}_0, \Gamma_0 \Rightarrow \Delta_0$

(Ind) If every topmost sequent is an initial sequent, the construction ends. Otherwise, for each non-initial sequent S_n, define:

- S_{n+1} is the result of applying all the prop. rules, $\Box_{\mathrm{L}}$, $\Diamond_{\mathrm{R}}$, ref, sym, tr and Euc to each pair of relational atoms in S_n;
- S_{n+2} is the result of applying all instances of $\Box_{\mathrm{R}}$ and $\Diamond_{\mathrm{L}}$ to formulas in S_{n+1};
- S_{n+3} is the result of applying all instances of ser to pairs of relational atoms in S_{n+2}.
- Repeat (Ind)

In the construction, apply each rule at most once to every formula / pair of formulas in a branch, e.g.:

Do not apply rule $\rightarrow_{\mathrm{L}}$ to sequent $xRy, x : A \rightarrow B \Rightarrow x : A$

Constructing the (infinite) countermodel

If $\mathcal{T}$ is infinite, then it has an infinite branch $\mathcal{B}^\times = (S_i)_{i<\omega}$. We construct a countermodel $\mathcal{M}^\times$ as follows:

- $W^\times = \{x \mid x \text{ occurs in } \mathcal{B}^\times\}$
- $xR^\times y$ iff xRy occurs in $(\mathcal{R}_i)_{i<\omega}$
- $v^\times(p) = \{x \mid x : p \text{ occurs in } (\Gamma_i)_{i<\omega}\}$

It is easy to verify that $\mathcal{M}^\times$ satisfies the frame conditions $\mathcal{X}$.

Truth Lemma. Take $\rho^\times(x) = x$, for each x in $(S_i)_{i<\omega}$. Then:

- If $x : A \in (\Gamma_i)_{i<\omega}$, then $\mathcal{M}^\times, \rho^\times \models x : A$
- If $x : A \in (\Delta_i)_{i<\omega}$, then $\mathcal{M}^\times, \rho^\times \not\models x : A$

Therefore, $\mathcal{M}^\times, \rho^\times \not\models S_0$, so S_0 is not valid.

Sources of non-termination, 1

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{x_2 R x_3, x_1 R x_2, x_0 R x_1 \Rightarrow x_0 : p}\,\text{ser}}{x_1 R x_2, x_0 R x_1 \Rightarrow x_0 : p}\,\text{ser}}{x_0 R x_1 \Rightarrow x_0 : p}\,\text{ser}}{\Rightarrow x_0 : p}\,\text{ser}$$

Sources of non-termination, 2

$$
\vee_R\,\dfrac{\Box_R\,\dfrac{\Diamond_R\,\dfrac{\Box_R\,\dfrac{\text{tr}\,\dfrac{\Diamond_R\,\dfrac{\Box_R\,\dfrac{\text{tr}\,\dfrac{\Diamond_R\,\dfrac{\Box_R\,\dfrac{\vdots}{xRy_0, y_0Ry_1, xRy_1, y_1Ry_2, xRy_2 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_2 : p, y_2 : \Box p}}{xRy_0, y_0Ry_1, xRy_1, y_1Ry_2, xRy_2 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_2 : p}}{xRy_0, y_0Ry_1, xRy_1, y_1Ry_2 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_2 : p}}{xRy_0, y_0Ry_1, xRy_1 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_1 : \Box p}}{xRy_0, y_0Ry_1, xRy_1 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p}}{xRy_0, y_0Ry_1 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p}}{xRy_0 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_0 : \Box p}}{xRy_0 \Rightarrow x : \Diamond\Box p, y_0 : \bot}}{\Rightarrow x : \Diamond\Box p, x : \Box\bot}}{\Rightarrow x : \Diamond\Box p \vee \Box\bot}
$$

Can we bound proof search?

☞ [Negri, 2005]: Minimality argument for some logics in the S5-cube (K, T, S4, S5)

☞ [Dyckhoff and Negri, 2012]: Termination for intermediate logics

Termination. For all the logics in the S5-cube, root-first proof search comes to an end in a finite number of steps.

- Saturated sequent in a branch
- After a finite number of steps, we reach either an initial sequent or a saturated sequent
- The finite branch of a saturated sequent provides a finite countermodel.

Exhaustive (finite) search tree: case of **labK4**

When building a branch $\mathcal{B}$ of a search tree for $S_0 = \mathcal{R}_0, \Gamma_0 \Rightarrow \Delta_0$:

- Do not apply a rule to (pairs of) formulas in S if S already satisfies the saturation condition associated to the rule;
- Apply $\Box_R$ and $\Box_L$ after all the other rules have been applied.

Some saturation conditions for $\mathcal{R}_i, \Gamma_i \Rightarrow \Delta_i$ along a branch:

(Tr) If xRy and yRz are in $\downarrow \mathcal{R}_i$, then xRz is in $\downarrow \mathcal{R}_i$;

($\wedge_L$) If $x : A \wedge B$ is in $\downarrow \Gamma_i$, then both $x : A$ and $x : B$ are in $\downarrow \Gamma_i$;

($\wedge_R$) If $x : A \wedge B$ is in $\downarrow \Delta_i$, then either $x : A$ or $x : B$ is in $\downarrow \Delta_i$;

($\Box_L$) If $x : \Box A$ is in $\downarrow \Gamma_i$ and xRy is in $\downarrow \mathcal{R}_i$, then $y : A$ is in $\downarrow \Gamma_i$

($\Box_R$) If $x : \Box A$ is in $\downarrow \Delta_i$, then for some y, xRy is in $\downarrow \mathcal{R}_i$ and $y : A$ is in $\downarrow \Delta_i$

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$\mathcal{R}_i, \Gamma_i \Rightarrow \Delta_i$ is saturated if it is not an initial sequent and it satisfies all the conditions.

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$(\Box_L)$ If $x : \Box A$ is in $\downarrow \Gamma_i$ and xRy is in $\downarrow \mathcal{R}_i$, then $y : A$ is in $\downarrow \Gamma_i$

$(\Box_R)$ If $x : \Box A$ is in $\downarrow \Delta_i$, then for some y, xRy is in $\downarrow \mathcal{R}_i$ and $y : A$ is in $\downarrow \Delta_i$ or
for some $z \neq x$, zRx is in $\downarrow \mathcal{R}_i$ and for all $x : B$, $x : B$ is in $\downarrow \Gamma_i / \downarrow \Delta_i$ iff $x : B$ is in $\downarrow \Gamma_i / \downarrow \Delta_i$;

$\mathcal{R}_i, \Gamma_i \Rightarrow \Delta_i$ is saturated if it is not an initial sequent and it satisfies all the conditions.

Example

$(\Box_R)$ If $x : \Box A$ is in $\downarrow \Delta_i$, then for some y, xRy is in $\downarrow \mathcal{R}_i$ and $y : A$ is in $\downarrow \Delta_i$ or
for some $z \neq x$, zRx is in $\downarrow \mathcal{R}_i$ and for all $x : B$, $x : B$ is in $\downarrow \Gamma_i / \downarrow \Delta_i$ iff $x : B$ is in $\downarrow \Gamma_i / \downarrow \Delta_i$;

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$$\Diamond_R\ \dfrac{xRy_0, y_0Ry_1, xRy_1, y_1Ry_2, y_0Ry_2, xRy_2 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_2 : p, y_2 : \Box p}{xRy_0, y_0Ry_1, xRy_1, y_1Ry_2, y_0Ry_2, xRy_2 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_2 : p}$$

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Finite countermodel construction: example

$$\text{fail}\ \overline{}$$

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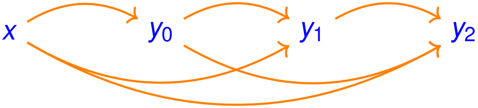

$x \nVdash \Box\bot$

$x \nVdash \Diamond\Box p$ $\quad y_0 \nVdash \Box p$ $\quad y_1 \nVdash \Box p$ $\quad y_1 \nVdash p$ $\quad y_2 \nVdash p$ $\quad y_2 \Vdash \Box p$

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$$
\begin{array}{ll}
\text{fail} & \overline{xRy_0, y_0Ry_1, xRy_1, y_1Ry_2, y_0Ry_2, xRy_2 \Rightarrow x : \Diamond\Box p, y_0 : \bot, y_1 : p, y_2 : p, y_2 : \Box p} \\
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\end{array}
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Summing up

Termination. Root-first proof search in **labK4** comes to an end along each branch $\mathcal{B}$ in a finite number of steps, with every leaf occupied by either an initial sequent or a saturated sequent.

Cut-free completeness (semantically). If $\models_{\mathcal{K}4} A$ then $\vdash_{\mathbf{labK4}} \Rightarrow x : A$.

Summing up

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Corollary. K4 has the finite model property.

Summing up

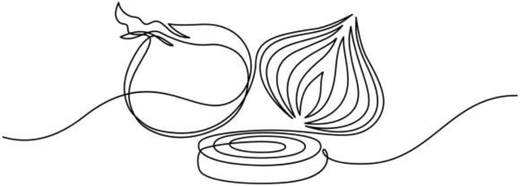
Termination. Root-first proof search in **labK4** comes to an end along each branch $\mathcal{B}$ in a finite number of steps, with every leaf occupied by either an initial sequent or a saturated sequent.

Cut-free completeness (semantically). If $\models_{\mathcal{K}4} A$ then $\vdash_{\mathbf{labK4}} \;\Rightarrow x : A$.

Corollary. K4 has the finite model property.

Corollary. The validity problem of K4 is decidable.

Neighbourhood semantics
for conditional logics



Conditional logics

1960-70: [Stalnaker], [Lewis], [Nute], [Chellas], [Burgess] ...

$$A, B \ ::= \ p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\Box A := \neg A > \bot \qquad \Diamond A := \neg(\bot > A)$$

- *If I hadn't overslept, then I would have caught the train.*
- *If Alice saw a lunar eclipse, then she would no longer believe that Earth is flat.*
- *If Tux is a bird then it can normally fly.*
- ...

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- *If Alice saw a lunar eclipse, then she would no longer believe that Earth is flat.*
- *If Tux is a bird then it can normally fly.*
- ...

☞ Neighbourhood models: To each world is associated a set of sets of worlds, used to interpret $A > B$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, v \rangle$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, v \rangle$

y

x

z

k

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, v \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$

y

x

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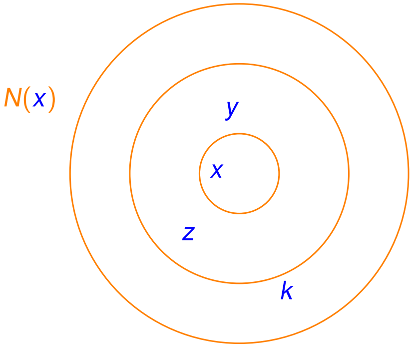

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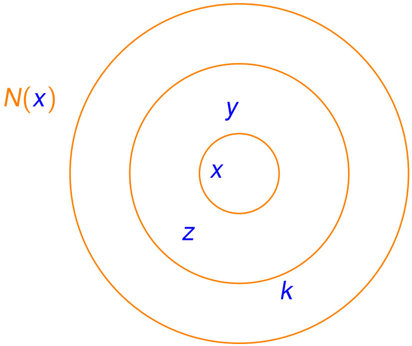

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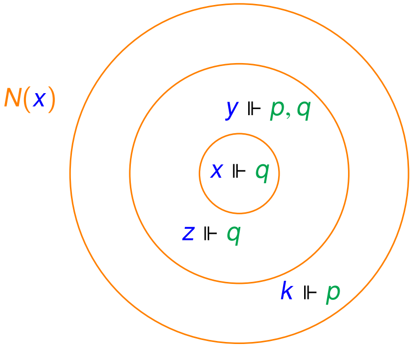

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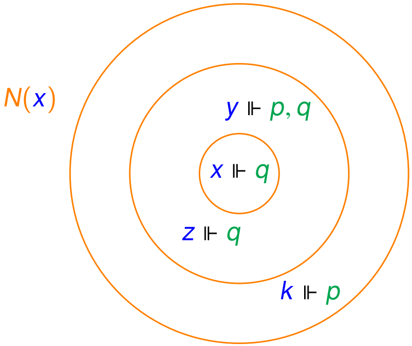
Nesting for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

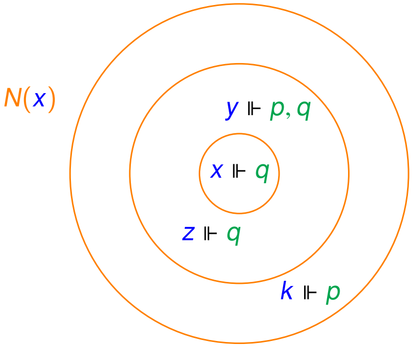


Neighbourhood models for VC

$\mathcal{M} = \langle W, N, v \rangle$ $N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $v : Atm \to \mathcal{P}(W)$

Nesting   for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering   for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



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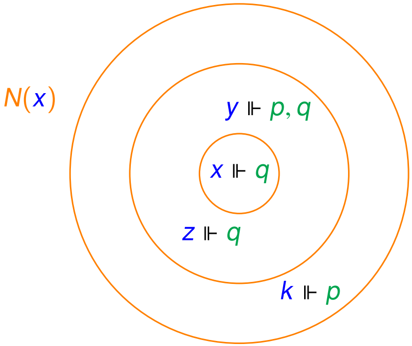

$x \Vdash p > q$ *iff* if there is $\alpha \in N(x)$ s.t. $\alpha \Vdash^{\exists} p$,
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} p$ and $\beta \Vdash^{\forall} p \to q$

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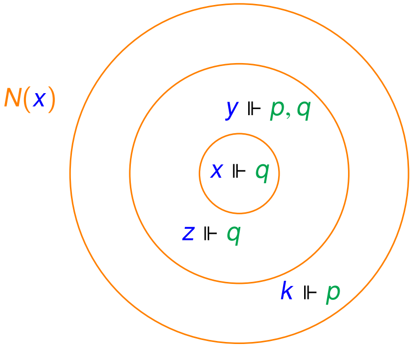

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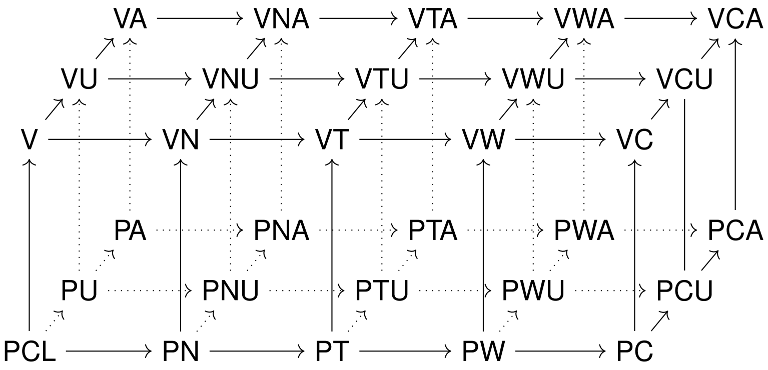
$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$ $\qquad \alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

Conditional logics



Nes for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

N $N(x) \neq \emptyset$

T there is $\alpha \in N(x)$ s.t. $x \in \alpha$

W $N(x) \neq \emptyset$ and for all $\alpha \in N(x)$, $x \in \alpha$

C $\{x\} \in N(x)$ and for all $\alpha \in N(x)$, $x \in \alpha$

U for all x, y, $\bigcup N(x) = \bigcup N(y)$

A for all x, y, $N(x) = N(y)$

Labelled calculi: enriching the language I

☞ Countably many variables for worlds $x, y, z, \ldots$

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- ☞ Some rules for frame conditions

C for all x, $\{x\} \in N(x)$ and for all $\alpha \in N(x)$, $x \in \alpha$

$$\text{C}\ \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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$$\text{Repl}_1\ \frac{y \in \{x\}, At(y), At(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Repl}_2\ \frac{y \in \{x\}, At(x), At(y), \mathcal{R}, \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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(inspired from [Negri, 2005])

Labelled calculi: enriching the language II

- ☞ Labelled formulas
  - ▹ $x : A \rightsquigarrow$ “x satisfies A”

Labelled calculi: enriching the language II

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- $x : A \rightsquigarrow$ "x satisfies A"
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- $a \Vdash^{\forall} A \rightsquigarrow$ "A is satisfied at all worlds of a"
- $x \Vdash_a A \mid B \rightsquigarrow$ "there is a $b \in N(x)$ such that $b \subseteq a$, $b \Vdash^{\exists} A$ and $b \Vdash^{\forall} A \rightarrow B$"

$$x \Vdash A > B \quad \textit{iff} \quad \text{if there is } \alpha \in N(x) \text{ s.t. } \alpha \Vdash^{\exists} A,$$
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- Some rules for $>$

$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A > B} \text{(a!)} \qquad \Vdash^{\exists}_L \frac{\mathcal{R}, x \in a, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta} \text{(x!)}$$

$$\Vdash_L \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \rightarrow B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \mathcal{R}, \Gamma \Rightarrow \Delta} \text{(a!)}$$

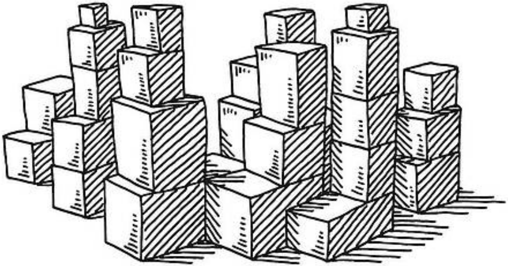
Summing up

☞ In [G, Negri and Olivetti, 2021]: modular labelled calculi for all the logics in the conditional lattice

☞ Other results: cut-admissibility for all the calculi; termination for almost all the logics

☞ For the logics in the lower part of the conditional lattice (without nesting), there are no known non-labelled analytic proof systems

(Bi-)Relational semantics
for intuitionistic (modal) logics



Intuitionistic logic

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \supset B$$

$$\neg A := A \supset \bot$$

$\mathcal{M} = \langle W, \leq, v \rangle$, where

- $W \neq \emptyset$
- $\leq$ is reflexive and transitive
- $v : Atm \to \mathcal{P}(W)$ s.t. if $x \leq y$ and $x \Vdash p$, then $y \Vdash p$

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$$x \Vdash A \supset B \quad \text{iff} \quad \text{for all } y \text{ s.t. } x \leq y, \text{ if } y \Vdash A, \text{ then } y \Vdash B$$

Intuitionistic logic

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$$\neg A := A \supset \bot$$

$\mathcal{M} = \langle W, \leq, v \rangle$, where

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- $\leq$ is reflexive and transitive
- $v : Atm \rightarrow \mathcal{P}(W)$ s.t. if $x \leq y$ and $x \Vdash p$, then $y \Vdash p$

$x \Vdash A \supset B$ iff for all y s.t. $x \leq y$, if $y \Vdash A$, then $y \Vdash B$

Monotonicity if $x \leq y$ and $x \Vdash A$, then $y \Vdash A$

Labelled calculus for intuitionistic logic

[Dyckhoff and Negri, 2011]

☞ Relational atoms and labelled formulas

- $x \leq y \rightsquigarrow$ "y is accessible from x in the preorder"
- $x : A \rightsquigarrow$ "x satisfies A"

Labelled calculus for intuitionistic logic

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$$\text{Ref}\ \frac{x \leq x, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Tr}\ \frac{x \leq z, x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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$$\text{init}\ \frac{}{x \leq y, \mathcal{R}, x : p, \Gamma \Rightarrow \Delta, y : p} \qquad \supset_R \frac{x \leq y, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta, y : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \supset B}\ (y!)$$

$$\supset_L \frac{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta, y : A \quad x \leq y, \mathcal{R}, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta}$$

$$\text{Ref}\ \frac{x \leq x, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Tr}\ \frac{x \leq z, x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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$$\supset_L \frac{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta, y : A \quad x \leq y, \mathcal{R}, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta}$$

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☞ Termination [Negri, 2014]

Intuitionistic modal logics

$$\mathcal{L} ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A \qquad \neg A \equiv A \supset \bot$$

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nec if A is provable, so is $\Box A$

k1 $\Box(A \supset B) \supset (\Box A \supset \Box B)$

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k1 $\Box(A \supset B) \supset (\Box A \supset \Box B)$

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k3 $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$

k4 $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

k5 $\Diamond \bot \supset \bot$

IK

Intuitionistic modal logics

$\mathcal{L} ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A \qquad \neg A \equiv A \supset \bot$

nec	if A is provable, so is $\Box A$	
k1	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	
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k3	$\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$	
k4	$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$	
k5	$\Diamond \bot \supset \bot$	
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t	$\Box A \supset A$	$A \supset \Diamond A$
b	$A \supset \Box\Diamond A$	$\Diamond\Box A \supset A$
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IS4 IS5

IT ITB

ID4 ID45

ID5

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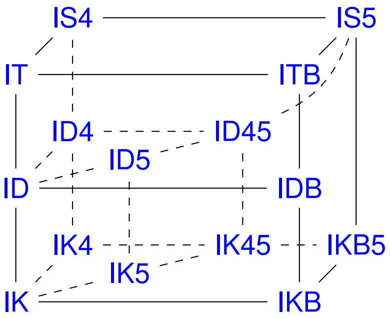

Bi-relational models for IK (and extensions)

- [Fisher Servi, 1984], soundness and completeness proof
- [Simpson, 1994]

$$\mathcal{M} = \langle W, R, \leq, v \rangle$$

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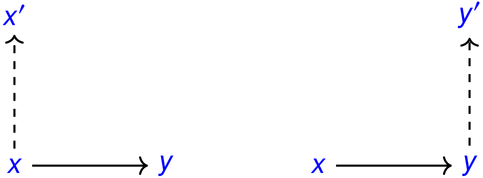

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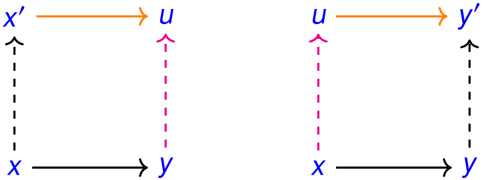

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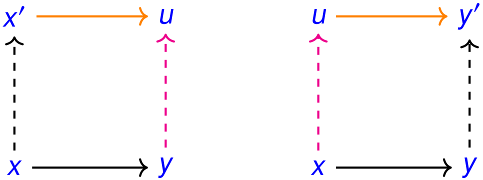

$x \Vdash A \supset B$	iff	for all y s.t. $x \leq y$, if $y \Vdash A$, then $y \Vdash B$
$x \Vdash \Box A$	iff	for all y, z s.t. $x \leq y$ and yRz, $z \Vdash A$
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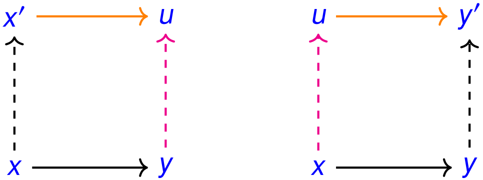

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☞ *Monotonicity* if $x \leq y$ and $x \Vdash A$, then $y \Vdash A$

Labelled calculi for IK (and extensions)

[Marin, Morales and Straßburger, 2021]

☞ Relational atoms and labelled formulas

- $x \leq y \rightsquigarrow$ "y is accessible from x in the preorder"
- $xRy \rightsquigarrow$ "y is accessible from x"
- $x : A \rightsquigarrow$ "x satisfies A"

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$$\square_L \frac{x \leq y, xRy, \mathcal{R}, x : \square A, z : A \Gamma \Rightarrow \Delta}{x \leq y, xRy, \mathcal{R}, x : \square A, \Gamma \Rightarrow \Delta} \qquad \square_R \frac{x \leq y, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta, z : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \square A} \text{ (y,z!)}$$

$$\text{F1} \frac{x'Ru, y \leq u, x \leq x', xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq x', xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \text{ (u!)}$$

$$\text{F2} \frac{x \leq u, uRy', xRy, y \leq y', \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, y \leq y', \mathcal{R}, \Gamma \Rightarrow \Delta} \text{ (u!)}$$

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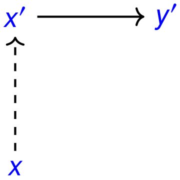

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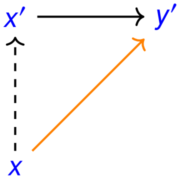

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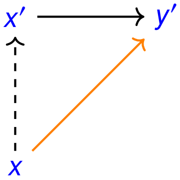

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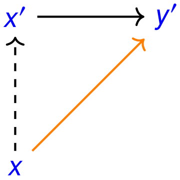

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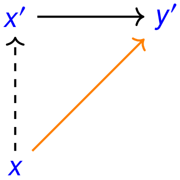

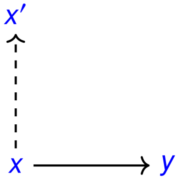

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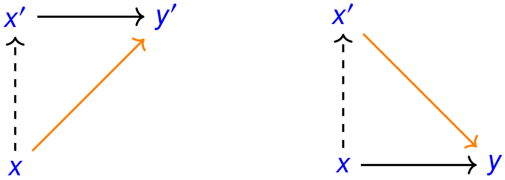

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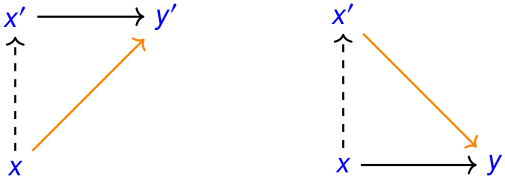

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Main references

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- Simpson, The proof theory and semantics of intuitionistic modal logic, PhD thesis, University of Edinburgh 1994.

Axiom systems, conditional logics (I)

PCL

Axiomatization classical propositional logic plus

(RCEA) $\dfrac{(A > C) \leftrightarrow (B > C)}{A \leftrightarrow B}$

(RCK) $\dfrac{(C > A) \rightarrow (C > B)}{A \rightarrow B}$

(R-And) $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

(ID) $A > A$

(CM) $(A > B) \wedge (A > C) \rightarrow ((A \wedge B) > C)$

(RT) $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

(OR) $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

V

Axiomatization of V plus

(CV) $(A > C) \wedge \neg(A > \neg B) \rightarrow ((A \wedge B) > C)$

Axiom systems, conditional logics (II)

Axioms for extensions

(N)	$\neg(\top > \bot)$	*Normality*
(T)	$A \to \neg(A > \bot)$	*Total reflexivity*
(W)	$(A > B) \to (A \to B)$	*Weak centering*
(C)	$(A \land B) \to (A > B)$	*Strong centering*
(U_1)	$(\neg A > \bot) \to (\neg(\neg A > \bot) > \bot)$	*Uniformity (1)*
(U_2)	$\neg(A > \bot) \to ((A > \bot) > \bot)$	*Uniformity (2)*
(A_1)	$(A > B) \to (C > (A > B))$	*Absoluteness (1)*
(A_2)	$\neg(A > B) \to (C > \neg(A > B))$	*Absoluteness (2)*

In the literature

☞ To each world is associated a set of sets of worlds, used to interpret modalities

Neighbourhood semantics

- As a semantics for non-normal modal logics [Scott, 1970], [Montague, 1970]
- As a semantics for belief revision [Grove, 1988]
- As a semantics for conditional logics: [Marti and Pinosio, 2013], [Negri & Olivetti, 2015], [Pacuit, 2017], [G, Negri and Olivetti, 2022]

Labelled calculi based on neighbourhood semantics

- Non-normal modal logics: [Negri, 2017]
- Conditional logics: [G, Negri and Olivetti, 2022]